Popularity in Three-Dimensional Stable Marriage

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Abstract

The relationship between stability and popularity has been well studied in the context of classical stable marriage and its variants. In these settings, all stable matchings are popular, and all strongly popular matchings are stable. Additionally, there are efficient algorithms for popularity testing and finding strongly popular matchings. This project investigates these properties of popularity in three-dimensional variants of stable marriage. We show that stable matchings need not be popular, and strongly popular matchings need not be stable. We also give graph-theoretic formulations of popularity testing and prove the hardness of deciding whether a popular matching exists in instances of certain variants of three-dimensional stable marriage.

1 Introduction

The interest in matching problems with preferences in computer science began with the Gale and Shapley seminal paper, where they defined the *Stable Marriage Problem* (SM) [5]. An instance of SM involves sets of n men and n women, where each has a strict preference list over the agents in the opposite set. The extension of SM addressed this paper consists of matching three sets of agents instead of two. We refer to these sets as men, women, and dogs, and each agent of a set has preferences over the *combinations* of the other two sets [11]. For example, each man has a strict ranking over all possible women-dog pairs. This is called *Three-Dimensional Stable Marriage* (3DSM). Work related to 3DSM has produced largely negative results, as finding a matching, even in many restricted cases, is NP-complete [2, 7, 8, 11].

The popularity criterion has also been investigated in instances of SM. First, Gärdenfors found that all strongly popular matchings are stable, and all stable matchings are popular [6]. More recently, it was shown that there are efficient algorithms for popularity testing and for finding popular matchings in instances of SM [1, 9]. In this paper, we determine which of these results hold in 3DSM and its variants and present results on popularity testing and the problem of deciding the existence of popular matchings in a given instance.

2 Problem Statements and Notation

2.1 Stability

The *Stable Marriage Problem With Incomplete Lists* (SMI) consists of two disjoint sets $\mathcal{M} = \{m_1, \ldots, m_n\}$ and $\mathcal{W} = \{w_1, \ldots, w_n\}$, referred to as *men* and *women* respectively. Let

$E \subseteq \mathcal{M} \times \mathcal{W}$ be the set of *acceptable* man-woman pairs. Each m_i strictly ranks the women in E. This ranking is called a *preference list*. For women w_j, w_k and man m_i, we say that m_i *prefers* w_j to w_k if $(m_i, w_j), (m_i, w_k) \in E(m)$ and w_j exceeds w_k on m_i's preference list. The same holds for women with respect to men. A *matching* is a set $M \subseteq E$ such that each man and woman appear in a pair at most once, and for some agent a, we denote $M(a)$ as a's partner in M. An acceptable pair $(m, w) \in E \setminus M$ *blocks* (or is called a *blocking pair* for) a matching M if:

1. m is unassigned or prefers w to $M(m)$, and

2. w is unassigned or prefers m to $M(w)$.

The matching M is stable if it admits no blocking pair.

The stable marriage problem SM is the special case of SMI where $E = \mathcal{M} \times \mathcal{W}$. Since all man-woman pairs are acceptable, when we define a matching in an instance of SM, it will always be a perfect matching (one where every agent is assigned exactly one partner). We will do the same for all settings that have no further restrictions on the acceptable assignments. Lastly, pairs (and later triples) will be represented with the agents concatenated instead of in a tuple form, so for some $m \in \mathcal{M}$ and $w \in \mathcal{W}$ we write mw to denote (m, w).

2.2 Stability in Three Dimensions

An instance of the *Three-Dimensional Stable Marriage Problem* (3DSM) of size n consists of disjoint sets $\mathcal{M} = \{m_1, \dots, m_n\}$, $\mathcal{W} = \{w_1, \dots, w_n\}$, and $\mathcal{D} = \{d_1, \dots, d_n\}$ of *men*, *women*, and *dogs* respectively. Each man m_i has a strict preference list over the pairs in $\mathcal{W} \times \mathcal{D}$. Similarly, each woman has preferences over $\mathcal{M} \times \mathcal{D}$, and each dog has preferences over $\mathcal{M} \times \mathcal{W}$. We define the $\prec$ relation for preferences such that m_i prefers $w_j d_k$ to $w_k d_\ell$ if an only if $w_k d_\ell \prec_{m_i} w_j d_k$. A *matching* M is a set of n disjoint triples. As in SM, we define a *blocking triple* as an element $t \in \mathcal{M} \times \mathcal{W} \times \mathcal{D}$ such that each agent a of t prefers their assignment in t to their assignment in M. A matching is *stable* if it admits no blocking triple.

In one special case of 3DSM, every man is primarily interested in the women, every woman is primarily interested in the men, and there are no restrictions on the preferences of dogs. For example, if some man m_i prefers $w_j d_k$ to $w_\ell d_m$, then for any $p, q \in [n]$ the man m_i prefers $w_j d_p$ to $w_\ell d_q$. We say that an instance of 3DSM with these restrictions has *lexicographic preferences* [4]. Interestingly, instances of 3DSM with lexicographic preferences always admit a stable matching, and there is an algorithm to find one in polynomial time [3].

We define a variant of 3DSM where ties are allowed in the agents' preference lists. Instead of agents ranking elements of combinations of the other two sets, they now rank disjoint subsets of those combinations For example, some m_i could have the preference list

$$m_i \;\big|\; \{w_1 d_1\} \quad \{w_2 d_2, w_3 d_3, \dots, w_{n-1} d_{n-1}\} \quad \{w_n d_n\},$$

where $w_1 d_1$ is his first choice, all of $w_2 d_2, \dots, w_{n-1} d_{n-1}$ is his second choice, and $w_n d_n$ is his last choice. Naturally, he is indifferent between every element of his second choice set. We call this setting *Three-Dimensional Stable Marriage with Ties* (3DSMT). When the preferences of an agent in instances of 3DSMT have a set with only one element, we will present the element without brackets on the preference list. For example, m_i's preference list may be written as

$$m_i \mid w_1 d_1 \quad \{w_2 d_2, w_3 d_3, \ldots, w_{n-1} d_{n-1}\} \quad w_n d_n.$$

In another variant of 3DSM, every agent has *consistent* preferences. In 3DSM, it is possible for some man m_i to rank $w_j d_k$ higher than $w_\ell d_k$ yet also rank $w_\ell d_p$ higher than $w_j d_p$. An instance where some agent's preference lists has this property is called *inconsistent*. To enforce consistency, Huang constructed a new kind of preference structure, where each agent has two *simple lists* that rank agents from the other two sets [7]. For example, each man m_i would have a simple list with strict preferences over $\mathcal{W}$ and another over $\mathcal{D}$. Then, these simple lists are combined to make a *preference poset* as shown in the following example. For man m_i and pairs $w_j d_k$ and $w_\ell d_p$, we have that $w_j d_k$ precedes $w_\ell d_p$ on m_i's preference poset only if w_j ranks at least as high as w_ℓ and d_k at least as high as d_p in the simple lists. Otherwise, they are incomparable.

A broader example of such a scheme is the *Precedence by Ordinal Number* (PON) scheme [7]. In PON, each agent strictly ranks the agents of a single set for each simple list. The preference poset is simply constructed by summing the ranks of the simple lists of the elements in each pair. For some man m_i and woman w_j, we define $\text{rank}(m_i, w_j)$ to be the position of w_j on m_i's simple list, where the most preferable pair is rank n and the least preferable is rank 1. Similarly, for a dog d_k, $\text{rank}(m_i, d_k)$ is the rank of d_k on m_i's preference list. This is defined analogously for the women and dogs. We define the partial order on the preference poset such that for any $i, j, k, \ell, p \in [n]$,

$$(w_j, d_k) \preceq_{m_i} (w_\ell, d_p)$$

is equivalent to

$$\text{rank}(m_i, w_j) + \text{rank}(m_i, d_k) \leq \text{rank}(m_i, w_\ell) + \text{rank}(m_i, d_p).$$

This can be generalized to a setting where agents give *ratings* instead of rankings, where a higher rating is more preferable [7]. This means that agents are able to provide the same rating (causing a tie) to two or more agents on their simple list. We call this setting *Precedence by Ordinal Number with Rankings* (PON-RATE). In this paper, we allow ratings of any positive integer value.

In these settings, agents can be indifferent between two different pairs of partners, which allows for a variety of "strengths" of stability. For a triple t and agent a, let $t(a)$ be the pair that a is partnered with in t. Given an instance of PON and a triple t, we define different levels of stability based on their corresponding blocking triples.

- *Weakly Stable*: triple t is a blocking triple if for each $a \in t$, we have $M(a) \prec_a t(a)$ (this is equivalent to a blocking triple in 3DSM).

- *Strongly Stable Matching*: triple t is a blocking triple if there are distinct $a, b \in t$ such that $M(a) \prec_a t(a)$ and $M(b) \prec_b t(b)$, while for the final $c \in t$, $M(c) \preceq_c t(c)$.

- *Super-Stable Matching*: triple t is a blocking triple if for some $a \in t$ $M(a) \prec_a t(a)$, while for the remaining $b, c \in t$, $M(b) \preceq_b t(b)$ and $M(c) \preceq_c t(c)$.

- *Ultra-Stable Matching*: triple $t \notin M$ is a blocking triple if for each $a \in t$, $t(a) \preceq_a M(a)$.

The *Three Dimensional Matching Problem* (3DM) is an NP-complete decision problem that was used to show that the problem of deciding whether a given instance of 3DSM admits a stable matching is NP-complete [11]. A three-dimensional matching is defined as follows. Let A, B, and D be finite, disjoint sets and $T \subseteq A \times B \times D$. A set $M \subseteq T$ is a matching if all triples in M are disjoint. 3DM is the decision problem where the input is an integer k and $T \subseteq A \times B \times D$, and the problem is to determine whether there exists a matching $M \subseteq T$ such that $|M| \geq k$. This problem is still NP-complete when $k = |A| = |B| = |D|$ [10]. When we refer to 3DM, we will refer to this restriction.

2.3 Popularity in Stable Marriage

For an instance of 3DSMT and two matchings M and M', we define $P(M, M')$ to be the set of agents who strictly prefer M to M'. Matching M is *more popular* than M' if more agents prefer M to M' than prefer M' to M, meaning $|P(M, M')| > |P(M', M)|$. Recall that if ties are allowed, then agents can be assigned different partners in M and M' while still being indifferent between them (meaning that they are in neither $P(M', M)$ nor $P(M, M')$). Matching M is *popular* if there is no matching M' that is more popular than M, and M is *strongly popular* if it is more popular than every matching $M' \neq M$. We define

$$\Delta(M, M') = |P(M, M')| - |P(M', M)|,$$

so M is popular if only if for all matchings M' we have that $\Delta(M, M') \geq 0$, and M is strongly popular if and only if for all matchings $M' \neq M$ we have that $\Delta(M, M') > 0$. These definitions are applied in exactly the same way to SM, 3DSM, PON, and PON-RATE, based on the construction of the preferences for the agents.

The Gärdenfors [6] first describes aspects of the relationship between stability and popularity in SM.

Theorem 1. *For instances of* SM*, all stable matchings are popular, and all strongly popular matchings are stable.*

Next, Biró, Irving, and Manlove [1] present the following algorithmic result.

Theorem 2. *There exists a polynomial-time algorithm to test whether a matching M in an instance of* SM *is popular.*

Since stable matchings can be found in polynomial time and always exist in instances of SM, from Theorem 1 we find that popular matchings can also be found in polynomial time [5]. However, this reasoning cannot be applied to strongly popular matchings, which led Biró, Irving, and Manlove [1] to leave the complexity of finding a strongly popular matching as an open problem, which was addressed by Huang and Kavitha [9], leading to the following result.

Theorem 3. *There exists a polynomial-time algorithm to find a strongly popular matching in an instance of* SM *or determine that no such matching exist.*

2.4 Our Contributions

First, we determine whether Theorem 1 generalizes to 3DSM and its variants. We found that, except in some cases with few agents, there are instances that admit stable matchings that are not popular and instances that admit strongly popular matchings that are not stable. We prove that this is the case in 3DSM even with the restriction to lexicographic preferences. Furthermore, we show that this is the case in PON and PON-RATE, even with different levels of stability.

To address Theorem 2, we define POP-TEST-3DSM to be the decision problem "given matching M in an instance of 3DSM, is M popular?" We give two graph-theoretic formulations for popularity testing in 3DSM in order to aid in determining its complexity. Since Theorem 1 does not always hold in instances of 3DSM, we define POP-3DSM to be the decision problem "given an instance I of 3DSM, does there exist a popular matching in I?" The decision problems POP-3DSMT and POP-PON-RATE are defined analogously. Finally, we give a reduction from 3DM to POP-3DSMT to prove that it is NP-hard, further implying that POP-PON-RATE is NP-hard.

3 Structural Results

This section details the extent that stable matchings are popular and strongly popular matchings are stable in variants and restrictions of 3DSM. We show that for most conceivable settings it is not the case that all stable matchings are popular and all strongly popular matchings are stable.

3.1 Three-Dimensional Stable Marriage

Proposition 1. For an instance of 3DSM with size $n = 2$. If the instance admits a strongly popular matching M^*, then M^* is stable.

Proof. We prove the contrapositive. Since there are two agents per set, for an agent $a_i \in \mathcal{M} \cup \mathcal{W} \cup \mathcal{D}$ let $\bar{a_i}$ be the other agent of the same set, so $\bar{m}_1 = m_2$. Let M be a matching that is not stable. Let $m_i w_j d_k$ be a blocking triple for M. Then $M' = \{m_i w_j d_k, \bar{m}_i \bar{w}_j \bar{d_k}\}$ is at least as popular as M. This is because, by the definition of a blocking triple, m_i, w_j, and d_k prefer the blocking triple to M. Even if the rest ($\bar{m}_i$, $\bar{w}_j$, and $\bar{d_k}$) of the agents prefer M, there is an equal number of agents who prefer M to M' and who prefer M' to M. Thus M is not strongly popular, meaning that for instances where $n = 2$, all strongly popular matchings are stable. □

However, Proposition 1 does not extend past two agents per group. In fact, even under the restriction of lexicographic preferences, we can find such an example.

Lemma 1. *Let I be the instance described in Figure 1, then*

$$M = \{m_2 w_1 d_3, m_1 w_3 d_2, m_3 w_2 d_1\}$$

is strongly popular but not stable.

m_1	w_1d_1	w_1d_2	w_1d_3	w_3d_1	w_3d_2	w_3d_3	w_2d_1	w_2d_3	w_2d_2
m_2	w_1d_2	w_1d_1	w_1d_3	w_3d_2	w_3d_3	w_3d_1	w_2d_3	w_2d_2	w_2d_1
m_3	w_2d_1	w_2d_2	w_2d_3	w_3d_2	w_3d_3	w_3d_1	w_1d_3	w_1d_1	w_1d_2
w_1	m_1d_1	m_1d_3	m_1d_2	m_2d_1	m_2d_3	m_2d_2	m_3d_2	m_3d_3	m_3d_1
w_2	m_3d_1	m_3d_2	m_3d_3	m_2d_2	m_2d_3	m_2d_1	m_1d_2	m_1d_3	m_1d_1
w_3	m_3d_2	m_3d_1	m_3d_3	m_1d_2	m_1d_1	m_1d_3	m_2d_2	m_2d_1	m_2d_3
d_1	m_2w_3	m_1w_1	m_1w_3	m_3w_2	m_2w_2	m_3w_3	m_2w_1	m_3w_1	m_1w_2
d_2	m_3w_1	m_1w_3	m_3w_3	m_2w_2	m_1w_1	m_2w_1	m_1w_2	m_3w_2	m_2w_3
d_3	m_2w_1	m_3w_3	m_2w_2	m_1w_1	m_3w_1	m_2w_3	m_1w_3	m_1w_2	m_3w_2

Figure 1: An instance of 3DSM restricted to lexicographic preferences with a strongly popular matching that is not stable.

Proof. First we will show that M is strongly popular in I. Suppose for the sake of contradiction that there exists M' such that M is not more popular than M'. Let A be the set of agents that prefer M' to M

Case 1: Suppose $M \cap M = \emptyset$, then we must show that $|A| < 5$. We know that no agent who is matched with their first choice in M is in A, so $A \subseteq \{m_1, m_2, w_1, w_3, d_1, d_2\}$.

- If $m_2 \in A$, then either $m_2w_1d_2 \in M'$ or $m_2w_1d_1 \in M'$. If $m_2w_1d_2 \in M'$ then $w_1, d_2 \notin A$, so $m_2 \notin A$.
- Now we have that $A \subseteq \{m_1, w_1, w_3, d_1, d_2\}$. If $d_2 \in A$ then $m_3w_2d_2 \in M'$, so $w_1 \notin A$, meaning that M is more popular than M'. By our assumption $d_2 \notin A$.

Thus, $|A| \leq 4$, so if the matchings are disjoint then M is more popular than M'.

Case 2: Suppose $M \cap M' = \{t\}$. We must show that $|A| \leq 2$.

- If $t = m_2w_1d_3$ then $A \subseteq \{m_1, w_3, d_1, d_2\}$. Then if $d_2 \in A$ we have $m_3w_1d_1 \in M'$, which is a contradiction, so $d_2 \notin A$. Now, if $m_1 \in A$, then m_1 must be paired with w_3, but they cannot be paired and prefer M'
- If $t = m_1w_3d_2$, then $A \subseteq \{m_2, w_1, d_1\}$, but m_2 cannot be paired with w_1 or w_3 and still prefer M' to M.
- If $t = m_3w_2d_1$, then $A \subseteq \{m_1, m_2, w_1, w_3, d_2\}$.
  - If $m_2 \in A$ then $m_2w_1d_2 \in M'$. Then if $m_1w_3d_3 \in M'$ we have $M = M'$, so $A \subseteq \{m_1, w_1, w_3, d_2\}$.
  - If $w_3 \in A$ then w_3 is paired with m_3, so $w_3 \notin A$.
  - We have $A \subseteq \{m_1, w_1, d_2\}$. If $d_2 \in A$ then $m_3w_1d_2 \in M'$, which is false by the construction of t.

Thus M is strongly popular.

The matching M is not stable because $m_1w_1d_1$ is a blocking triple. $\square$

m_1	w_1d_1	w_1d_2	w_2d_2	w_2d_1
m_2	w_2d_2	w_2d_1	w_1d_1	w_1d_2
w_1	m_2d_2	m_2d_1	m_1d_2	m_1d_1
w_2	m_1d_1	m_1d_2	m_2d_1	m_2d_2
d_1	m_1w_2	m_2w_1	m_1w_2	m_2w_2
d_2	m_2w_1	m_2w_2	m_1w_1	m_1w_2

Figure 2: An instance of 3DSM restricted to lexicographic preferences with a stable matching that is not popular.

Likewise, we give an example of an instance with lexicographic preferences that admits a stable matching that is not popular.

Lemma 2. *For the instance of* 3DSM *described in Figure 2 and matching* $M = \{m_1w_1d_1, m_2w_2d_2\}$, *the matching* M *is stable but not popular.*

Proof. Let $M' = \{m_2w_1d_2, m_1w_2d_1\}$. Because in M each man m is paired with their first choice woman w and dog d, there cannot be a blocking triple, since that requires m to prefer another pair. Thus M is stable. Similarly, in M' every woman w' is paired with their first choice couple of m' and d', so there cannot be a blocking triple, meaning that M' is stable.

In M, each man is paired with his first choice, and each woman are paired with her last choice. Similarly, in M' each woman is paired with her first choice, while each man is paired with his last choice. Thus, when considering only the men and women, there are an equal number of agents who prefer the other matching. However, d_1 and d_2 both prefer M', since the couple they are paired with is their first choice in M' while it is not for M. Thus $P(M', M) = \{w_1, w_2, d_1, d_2\}$ while $P(M, M') = \{m_1, m_2\}$, so M is not popular. □

From Lemmas 1 and 2 we obtain the following result.

Theorem 4. *There are instances of* 3DSM*, even under the restriction lexicographic preferences, where the properties mentioned in Theorem 1 do not hold. Namely, there are instances that admit strongly popular matchings that are not stable, and there are instances that admit stable matchings that are not popular.*

3.2 Precedence by Ordinal Number

Because PON introduces stronger versions of stability, it is natural to investigate the relationship between popularity and these different notions of stability.

Lemma 3. *Let* I *be the instance of* PON *described in Figure 3, the matching*

$$M = \{m_2w_1d_2, m_1w_2d_1\}$$

is ultra stable but not popular.

m_1	w_1	w_2
	d_2	d_1
m_2	w_2	w_1
	d_1	d_2
w_1	m_1	m_2
	d_2	d_1
w_2	m_1	m_2
	d_1	d_2
d_1	m_1	m_2
	w_1	w_2
d_2	m_2	m_1
	w_1	w_2

Figure 3: An instance of PON that admits an ultra-stable matching that is not popular.

Proof. The men m_1 and m_2 are paired to their first-choice partners in M (of which there is only one combination of women and dogs), so for any $t = (m, w, d) \in (\mathcal{M} \times \mathcal{W} \times \mathcal{D}) \setminus M$, man m strictly prefers $M(m)$ to $t(m)$. Thus, M is ultra stable.

Consider $M' = \{m_2w_1d_1, m_1w_2d_2\}$. We have $P(M, M') = \{m_1, m_2\}$ while

$$P(M', M) = \{w_1, d_2, w_2\},$$

so M is not popular. □

Since we have shown there exists an ultra stable matching that is not popular, the same follows for weakly stable, strongly stable, and super-stable matchings.

Lemma 4. *For the instance of* PON *described in Figure 4. The matching*

$$M = \{m_1w_1d_1, m_2w_2d_2, m_3w_3d_3\}$$

is strongly popular but not weakly stable.

Proof. Let M' be a matching that is not M. Suppose M and M' are disjoint. Then

$$\{m_1, m_3, w_1, w_2, d_2, d_3\} \subseteq P(M, M'),$$

since those are the agents that are matched with their first choice partners in M. Thus, M is more popular than M'.

If M and M' have one triple in common, then $|P(M, M')| \geq 4$ since four agents not in the triple are paired with their first choice. Then $|P(M', M)| \leq 2$, so M is more popular than M'. Thus, M is strongly popular.

The matching M is not even weakly stable since $m_2w_3d_1$ is a blocking triple. □

We have shown that there is an instance of PON that admits a strongly popular matching that is not weakly stable, so we know that Lemma 4 holds for strongly stable, super-stable, and ultra-stable matchings as well.

m_1	w_3	w_2	w_1
	d_2	d_3	d_1
m_2	w_2	w_1	w_3
	d_3	d_1	d_2
m_3	m_1	w_2	w_3
	d_1	d_2	d_3
w_1	m_3	w_2	w_1
	d_2	d_3	d_1
w_2	m_1	w_3	w_2
	d_1	d_3	d_2
w_3	m_1	w_3	w_2
	d_3	d_2	d_1
d_1	m_1	m_3	m_2
	w_1	w_3	w_2
d_2	m_3	m_1	m_2
	w_3	w_1	w_2
d_3	m_2	m_1	m_3
	w_2	w_1	w_3

Figure 4: An instance of PON that admits a strongly popular matching that is not weakly stable.

Theorem 5. *There exists instances of* PON *where the properties mentioned in Theorem 1 do not hold, even when replacing traditional (weak) stability with strong, super, or ultra stability. That is, for each definition of stability, there are instances that admit strongly popular matchings that are not stable, and there are instances that admit stable matchings that are not popular.*

4 Algorithmic Results

The primary algorithmic problem we aim to solve is determining the hardness of POP-3DSM. For this reason, we also study the problem of popularity testing, called POP-TEST-3DSM, since if POP-TEST-3DSM is in P, then POP-3DSM is in NP. This is because a popular matching serves as a certificate for popularity, and the popularity test could be an efficient verifier. We show the equivalence of POP-TEST-3DSM to two graph-theoretic problems, and prove the hardness of POP-3DSMT, the generalized version of POP-3DSM that allows ties in preference lists.

4.1 Popularity Testing

Let M be a matching in an instance of 3DSM. Define $H_M = (V, E)$ to be a hypergraph such $V = \mathcal{M} \cup \mathcal{W} \cup \mathcal{D}$ and $E = \mathcal{M} \times \mathcal{W} \times \mathcal{D}$. For $e \in E$, let $wt(e)$ denote the weight of e. For

$a \in \mathcal{M} \cup \mathcal{W} \cup \mathcal{D}$ and b, a pair on a's preference list, let

$$\text{vote}_M(a,b) = \begin{cases} 1 & \text{if } a \text{ prefers } b \text{ to } M(a), \\ 0 & \text{if } b = M(a), \text{ and} \\ -1 & \text{if } a \text{ prefers } M(a). \end{cases}$$

Now, for each $e = (m, w, d) \in E$ we let $wt(e) = \sum_{a \in e} \text{vote}_M(a, e(a))$. For some matching $M' \subseteq E$, the weight of the matching in H_M is denoted $wt(M') = \sum_{t \in M'} wt(t)$. There is a

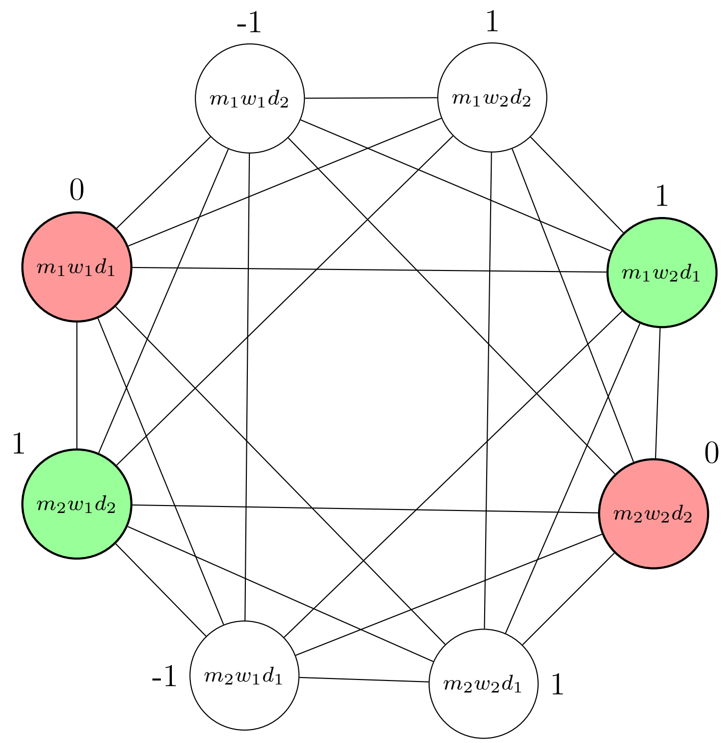

Figure 5: The graph G_M from the instance of 3DSM in Figure 2, with matchings M and M' colored red and green, respectively.

polynomial-time test for popularity in SMI [1]. We give a similar structure for popularity testing in 3DSM.

Theorem 6. *For an instance of* 3DSM *and hypergraph H_M, the matching M is popular in I if and only if M is a maximum-weight perfect matching of H_M.*

Proof. For any perfect matchings M_1 and M_2, we have $\Delta(M_1, M_2) = |P(M_1, M_2)| - |P(M_2, M_1)|$ and $\Delta(M_1, M_2) > 0$ if and only if M_1 is more popular than M_2. Let M' be a perfect matching in H_M. We have

$$\begin{aligned}
wt(M') &= wt(M' \setminus M) + wt(M' \cap M) \\
&= wt(M' \setminus M) \\
&= \sum_{a \in \mathcal{M} \cup \mathcal{W} \cup \mathcal{D}} \text{vote}_M(a, M'(a)) \\
&= |P(M', M)| - |P(M, M')| \\
&= \Delta(M', M),
\end{aligned}$$

so if $wt(M') > 0$, then M is not popular. On the other hand, if M is not popular, then there exists a perfect matching M^* such that $\Delta(M^*, M) > 0$, meaning that M is not a maximum-weight perfect matching. Thus, M is popular if and only if M is a maximum-weight perfect matching of H_M. □

It is unknown whether this problem is polynomial-time solvable, so we give another formulation for POP-TEST-3DSM on graphs.

Define $G_M = (V, E)$ such that $V = \mathcal{M} \times \mathcal{W} \times \mathcal{D}$ and $E = \{\{u, v\} : u \cap v \neq \emptyset, \forall u, v \in V\}$. For $v \in V$, let $wt(v) = \sum_{a \in v} \text{vote}(a, M(a))$ and for $M' \subseteq V$ let $wt(M') = \sum_{t \in M'} wt(t)$. Equivalently, V represents the edge set of H_M, and E connects non-disjoint triples.

Theorem 7. *A matching M in an instance of* 3DSM *is a maximum weight n-independent set of G_M if and only if M is popular.*

Proof. By the definition of a 3DSM matching, $|M| = n$ and M is an independent set in G_M. Let M' be an independent set of size n of G_M. The set M' is a matching because it assigns all agents exactly once. By the argument with H_M, we know that $wt(M') = \Delta(M', M)$, so $wt(M') > 0$ implies that M' is more popular than M. Then there is a matching M^* such that $\Delta(M^*, M) > 0$. Thus, in G_M we have that $wt(M^*) > 0$, so M is not a maximum-weight independent set of size n. □

As an example to explain the intuition behind the formulation, Figure 5 shows G_M for the instance of 3DSM in Figure 2, where

$$M = \{m_1 w_2 d_1, m_2 w_1 d_2\}$$

and

$$M' = \{m_1 w_2 d_1, m_1 w_2 d_1\}.$$

We can tell that M is not popular because M' is an independent set with weight greater than the weight of M in G_M.

4.2 NP-Hardness in the Case With Ties

We prove that POP-3DSMT is NP-hard by providing a polynomial reduction from 3DM. First, we discuss a motivating example that we use as a gadget. This instance has the interesting property that it admits no popular matching, which we use in our reduction.

Lemma 5. *The instance of* 3DSMT *described in Figure 6 does not admit a popular matching.*

m_1	w_1d_1	w_1d_2	w_2d_1	w_2d_2
m_2	w_1d_1	w_1d_2	w_2d_1	w_2d_2
w_1	m_1d_1	m_1d_2	m_2d_1	m_2d_2
w_2	m_2d_1	m_1d_2	m_1d_1	m_2d_2
d_1	m_2w_1	m_1w_2	m_1w_1	m_2w_2
d_2	m_2w_1	m_2w_2	m_1w_1	m_1w_2

Figure 6: An instance of 3DSMT with no popular matching.

Proof. Let

$$M_1 = \{m_1w_1d_1, m_2w_2d_2\}$$
$$M_2 = \{m_1w_1d_2, m_2w_2d_1\}$$
$$M_3 = \{m_1w_2d_1, m_2w_1d_2\}$$
$$M_4 = \{m_1w_2d_2, m_2w_1d_1\}$$

We have

- M_3 is more popular than M_1 since $P(M_3, M_1) = \{m_2, w_2, d_1, d_2\}$,
- M_1 is more popular than M_2 since $P(M_1, M_2) = \{m_1, w_1, d_1, d_2\}$,
- M_4 is more popular than M_3 since $P(M_4, M_3) = \{m_2, w_1, w_2, d_1\}$, and
- M_2 is more popular than M_4 since $P(M_2, M_4) = \{m_1, w_1, w_2, d_2\}$.

Since for every matching M there is an M' that is more popular than M, there is no popular matching. $\square$

$\vdots$						
a_i	$T(a_i)$	$(B \times D) \setminus T(a_i)$	$B \times \{\delta\}$	$\{\beta\} \times D$	$\beta\delta$	$\ldots$
$\vdots$						
$\vdots$						
b_i	$T(b_i)$	$(A \times D) \setminus T(b_i)$	$A \times \{\delta\}$	$\{\alpha\} \times D$	$\alpha\delta$	$\ldots$
$\vdots$						
$\vdots$						
d_i	$T(d_i)$	$\{\alpha\} \times B$	$A \times \{\beta\}$	$(A \times B) \setminus T(d_i)$	$\alpha\beta$	$\ldots$
$\vdots$						
α	$B \times D$	$B \times \{\delta\}$	$\{\beta\} \times D$	$\beta\delta$	$\ldots$	
β	$\{\alpha\} \times D$	$A \times \{\delta\}$	$A \times D$	$\alpha\delta$	$\ldots$	
δ	$\{\alpha\} \times B$	$\alpha\beta$	$A \times B$	$A \times \{B\}$	$\ldots$	

Figure 7: The mapping from instances of 3DM to instances of 3DSMT.

In order to understand how this example functions in our reduction, we first give the mapping between the two problems. To simplify notation, we briefly abandon using $\mathcal{M}$, $\mathcal{W}$, and $\mathcal{D}$ for the men, women, and dogs. For an instance $I = (A, B, D, T)$ of 3DM, we define I' to be an instance of POP-3DSMT with sets $A' = A \cup \{\alpha\}$, $B' = B \cup \{\beta\}$, and $D' = D \cup \{\delta\}$. As a result of our simplified notation, we make no distinction between elements of A', B', and D' and their corresponding elements in I. Finally for any $e \in A \cup B \cup D$, we let $T(e)$ be the set of partners of e in T, and "..." in the preference lists refers to a fixed but arbitrary preference of the remaining pairs.

Lemma 6. *For an instance $I = (A, B, D, T)$ of* 3DM*, let I' be the result of the mapping described in Figure 7 with corresponding sets A', B', and D'. If I admits a perfect matching M, then $M' = M \cup \{\alpha\beta\delta\}$ is a popular matching for I'.*

Proof. If M is perfect, then all of A, B, and D are matched to their first choice pair in I'. Suppose some matching M^* of I' does not match elements of A, B, and D corresponding to a perfect matching with respect to T. Then there exists $a \in A$, $b \in B$, and $c \in C$ that are matched to triples not in T, so a, b, and c prefer M' to M^*. Then, the only agents that could be better off in M^* are α, β, and δ. Thus, $\Delta(M', M^*) \geq 0$, meaning that M' is at least as popular as any matching that does not correspond to a perfect matching of M.

Suppose M^* does correspond to a perfect matching of I, meaning that all of A, B, and D are matched in the triples of T. Then all elements of A, B, and D are indifferent between M^* and M' since they are matched to their first choice in both. Also, α, β, and δ have the same partners in M^* as they do in M', so $\Delta(M', M^*) = 0$. □

Lemma 7. *For an instance I of* 3DM *and its corresponding* 3DSMT *instance I', if there is no perfect matching in T, then there is no popular matching in I'.*

Proof. We show that for each matching M^* that matches k triples in T for $0 \leq k \leq n-1$, there exists a matching M' that is more popular than M^*. Let M be a matching of size k of T and let $\overline{M}$ be an arbitrary matching of the elements of A', B', and D' that are not accounted for in M. Let $M^* = M \cup \overline{M}$. Then, following the motivating example:

1. If $\alpha\beta\delta \in M^*$, then for any $a_i b_j d_\ell \in (A \times B \times D) \setminus T$ we have $M' = (M^* \setminus \{\alpha\beta\delta, a_i b_j d_\ell\}) \cup \{a_i \beta d_\ell, \alpha b_j \delta\}$ is more popular than M^*.
2. If there are $i, j, \ell \in [n]$ such that $\alpha\beta d_\ell, a_i b_j \delta \in M^*$ then $M' = (M^* \setminus \{\alpha\beta d_\ell, a_i b_j \delta\}) \cup \{\alpha\beta\delta, a_i b_j d_\ell\}$ is more popular than M^*.
3. If there are $i, j, \ell \in [n]$ such that $a_i \beta d_\ell, \alpha b_j \delta \in M^*$ then $M' = (M^* \setminus \{a_i \beta d_\ell, \alpha b_j \delta\}) \cup \{a_i \beta \delta, \alpha b_j d_\ell\}$ is more popular than M^*.
4. If there are $i, j, \ell \in [n]$ such that $a_i \beta \delta, \alpha b_j d_\ell \in M^*$ then $M' = (M^* \setminus \{a_i \beta \delta, \alpha b_j d_\ell\}) \cup \{a_i b_j \delta, \alpha\beta d_\ell\}$ is more popular then M^*.

Therefore, a matching that contains k triples from T is not popular. If I does not admit a perfect matching, then M cannot match n elements of T, so every matching in I' is accounted for. □

Theorem 8. *The decision problem* POP-3DSMT *is NP-hard*

Proof. For any instance I of 3DM and the corresponding I' of 3DSMT, if there exists a perfect matching of T, then there is a popular matching in I'. Namely, for a perfect matching $M \subseteq T$, the matching $M' = M \cup \{\alpha\beta\delta\}$ is popular. Finally, we showed that if there is no perfect matching in I, then there is no popular matching in I'. Thus I admits a perfect matching if and only if I' admits a popular matching, so POP-3DSMT is NP-hard. $\square$

5 Concluding Remarks

We have shown that in 3DSM and its variants, certain properties of popularity from two-sided stable marriage do not hold. We have also given graph-theoretic formulations for popularity testing and have proven that POP-3DSMT is NP-hard. We intend to further investigate this formulation to better understand the complexity of both POP-TEST-3DSM and POP-3DSM. It is important to answer the question of whether there is a polynomial-time algorithm for popularity testing, since the existence of such an algorithm would mean that POP-3DSM is in NP. If popularity testing is NP-hard, then POP-3DSM may be NP-complete or strictly NP-hard.

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